Equivalence of linear response among extended optimal velocity models

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We investigate the property of extended optimal velocity (OV) models of traffic flow, in which a driver looks at arbitrary number of vehicles that precede. We prove an equivalence of linear response among extended models. This equivalence provides a natural understanding of the improvement of the stability of traffic flow.

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Over the past decade, many physicists have been interested in the traffic dynamics, especially traffic congestion. It is well known that free traffic flow on highway changes to congested flow and clusters of vehicles are formed when the vehicle density increases. To explain this phenomenon, a lot of studies have been done from the physical viewpoint [1–5]. The optimal velocity (OV) model [6,7] is one of such traffic models, which have successfully described the dynamical formation of traffic congestion. In the framework of the OV model, it is shown that the congestion appears in a certain condition, and that the reaction to the preceding vehicle plays an essential role to organize traffic congestion.

In the social viewpoint, the most important problem is how to suppress the appearance of traffic congestion. However, the only possibility to improve the stability of traffic flow in the original OV model is to take a large value of sensitivity, because a driver is supposed to look at the preceding vehicle only. In order to efficiently suppress the formation of congestion, it is necessary to extend the model, for example, incorporating the effect of watching other vehicles as well as the preceding vehicle. Several authors have shown that the stability of traffic flow is improved by the effect from other vehicles; a vehicle that follows [8–10], the next to the preceding vehicle [11,12], and many other vehicles [13].

In the previous paper we extended the OV model such that a driver looks at arbitrary number of vehicles that precede and follow [14]. Due to the extension, the stability of traffic flow is improved. The extended models have different features in the following two cases. One is the "forward looking" OV model which incorporates the effect from vehicles that precede and the other is the "backward looking" OV model which incorporates the effect from vehicles that follow. We found there exists a certain set of parameters which make traffic flow "most stable" in the former case.

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In this paper, we discuss the property of the forward looking OV model with the set of "most stable" parameters. We investigate the linear response to a disturbance on a uniform flow, and prove an equivalence of the linear response between the extended model and the OV model. The equivalence clarifies the reason why the extension improves the stability and also shows the limitation of stability.

The forward looking OV model with the parameter k_+ is formulated by the equation of motion

$$\frac{d^2x_n}{dt^2}=a\left[V(\Delta x_{n+k_+},\ldots,\Delta x_{n+1},\Delta x_n)-\frac{dx_n}{dt}\right], \quad (1)$$

where x_n is position of the nth vehicle and $\Delta x_{n+k}\equiv x_{n+k+1}-x_{n+k}$ for $k=k_+,k_+-1,\ldots,0$ are headway of the $(n+k)$th vehicle. Vehicles are numbered such that the $(n+1)$th vehicle precedes the nth vehicle. The model with $k_+=0$ is the original OV model. The extended OV function $\mathrm{V}(\Delta x_{n+k_+},\ldots,\Delta x_n)$ represents an optimal velocity of the nth vehicle. The parameter a, which has the dimension of inverse of time, is called sensitivity.

Model (1) has a solution of uniform flow

$$x_n=bn+\mathrm{V}(b,b,\ldots,b)t+\text{const}, \quad (2)$$

where all the vehicles have the same headway b and the same velocity $\mathrm{V}(b,\ldots,b)$.

Let y_n be a small fluctuation imposed on the uniform flow. From Eq. (1), $y_n(t)$ satisfies the linearized equation

$$\frac{d^2y_n}{dt^2}=a\left[\sum_{k=0}^{k=k_+} f_k\Delta y_{n+k}-\frac{dy_n}{dt}\right], \quad (3)$$

where $\Delta y_{n+k}=y_{n+k+1}-y_{n+k}$ and f_k is defined by

$$f_k=\frac{\partial}{\partial\Delta y_{n+k}}\mathrm{V}(b+\Delta y_{n+k_+},\ldots,b+\Delta y_n)\Big|_{\Delta y=0} \quad (4)$$

for $k=k_+,k_+-1,\ldots,0$.

We can calculate the stability condition for sensitivity a. If a is larger than a certain value decided by f_k, then uniform flow (2) is stable. In the previous paper, we have shown that the following choice of f_k,

$$f_{k_+}=f_{k_+-1}=f_{k_+-2}=\cdots=f_0=\frac{1}{k_++1}, \tag{5}$$

gives the minimum value of sensitivity a which makes the uniform flow stable [14].

In this paper we consider the model with parameters (5), and show an equivalence between this model and the original OV model with respect to the behavior of linear response to the disturbance. For this purpose, we define the following test function [14] and prove the function for the extended model is equal to that for the OV model:

$$F_{lm}(t)=\frac{1}{\epsilon^2}\sum_n \frac{d^l y_n(t)}{dt^l}\frac{d^m y_n(t)}{dt^m}, \tag{6}$$

where the indices l and m are arbitrary non-negative integers, and $\epsilon \sim O(y_n)$ is a normalization constant. The test function is a general bilinear function made from $y_n(t)$. For example, the function represents the fluctuation of position for $l=m=0$, and the fluctuation of velocity for $l=m=1$.

Linearized equation (3) with the set of parameters (5) leads to

$$\frac{d^2y_n}{dt^2}=a\left[\frac{1}{k_++1}(y_{n+k_++1}-y_n)-\frac{dy_n}{dt}\right]. \tag{7}$$

We assume the periodic boundary condition $x_N \equiv x_0$, where N is the total number of vehicles. Equation (7) has mode solutions

$$y_{n,\theta}=\exp[in\theta-i\omega(\theta)t], \tag{8}$$

where $\theta=2\pi j/N$, $j=0,1,2,\dots,N-1$. $\omega(\theta)$ is given by the solution of the algebraic equation

$$\omega(\theta)^2+ia\omega(\theta)+a\frac{1}{k_++1}\{\exp[i(k_++1)\theta]-1\}=0. \tag{9}$$

As a simple case, we consider the initial condition $y_n(0)=\epsilon\delta_{n0}$ and $\dot{y}_n(0)=0$. Then the solution of Eq. (7) becomes

$$y_n(t)=\frac{\epsilon}{N}\sum_{\theta,\sigma}\frac{\omega_{-\sigma}(\theta)}{\omega_{-\sigma}(\theta)-\omega_\sigma(\theta)}\exp[in\theta-i\omega_\sigma(\theta)t], \tag{10}$$

where $\sigma=\pm$ is an index of two solutions of Eq. (9).

Using this solution and carrying out the summation with respect to n, we can obtain $F_{lm}(t)$, which is expressed symbolically as

$$F_{lm}(t)=\frac{1}{N}\sum_\theta G(\omega(\theta),t). \tag{11}$$

We note that the function $G(\omega(\theta),t)$ does not depend explicitly on θ.

In the OV model, we remind that the algebraic equation for $\omega(\theta)$ corresponding to Eq. (9) is written as

$$\omega(\theta)^2+ia\omega(\theta)+af[\exp(i\theta)-1]=0, \tag{12}$$

where $f=V'(b)$. To investigate the equivalence between these models, we set

$$f=\frac{1}{k_++1}. \tag{13}$$

Denoting the solution of Eq. (12) by $\omega^{\mathrm{ov}}(\theta)$, the solution of Eq. (9) is expressed as

$$\omega(\theta)=\omega^{\mathrm{ov}}((k_++1)\theta). \tag{14}$$

Then test function (11) is rewritten as

$$F_{lm}(t)=\frac{1}{N}\sum_\theta G(\omega^{\mathrm{ov}}[(k_++1)\theta],t). \tag{15}$$

If N and k_++1 are mutually prime, $F_{lm}(t)$ reduces to the test function for the OV model.

$$F_{lm}(t)=\frac{1}{N}\sum_\theta G(\omega^{\mathrm{ov}}(\theta),t)\equiv F_{lm}^{\mathrm{ov}}(t). \tag{16}$$

In general, the summation for θ is turned to the integration in the limit $N\to\infty$. The test function is expressed as

$$F_{lm}(t)=\frac{1}{2\pi}\int_0^{2\pi}d\theta G(\omega^{\mathrm{ov}}[(k_++1)\theta],t) \tag{17}$$

$$=\frac{1}{2\pi(k_++1)}\int_0^{2(k_++1)\pi}d\phi G(\omega^{\mathrm{ov}}(\phi),t) \tag{18}$$

$$=\frac{1}{2\pi}\int_0^{2\pi}d\phi G(\omega^{\mathrm{ov}}(\phi),t) \tag{19}$$

$$=F_{lm}^{\mathrm{ov}}(t). \tag{20}$$

In the derivation from Eq. (18) to Eq. (19), we use the periodicity of the function $\omega(\phi)$ with respect to ϕ.

The equality of test functions is obtained for the case of the initial condition $y_n(0)=\epsilon\delta_{n0}$ and $\dot{y}_n(0)=0$, where a disturbance is added to only the first vehicle. For more general case where disturbances are added to all vehicles randomly, we can find the same result by taking an average over initial conditions. Thus we have proved the equivalence of linear response between the forward looking model with the set of "most stable" parameters (5) and the original OV model with parameter (13).

Suppose the OV model with $f=V'(b)=1$. The extended model has the same uniform flow solution as this model, and has the same linear response as the OV model with $f/(k_++1)$. Note that the stability condition of the uniform flow is given by $a>2f$ in the OV model. This condition indicates that a small f makes the traffic flow stable. Thus the improvement of the stability in the extended model is naturally understood by the equivalence to the OV model with a small f.

The "most stable" parameters and the equivalence mentioned above indicate the existence of a limitation in the improvement of the stability. We have shown that the stability of the extended model is understood within the framework of the original OV model. Therefore we can conclude that the traffic congestion is unavoidable even in the extended models as the same as the original OV model. In other words, the instability of traffic flow cannot be completely removed by the "forward looking" extension of the OV model. The instability of traffic flow always exists in a certain region of parameter space, which corresponds to the real situation where vehicles run with large velocity and small headway.

It is often considered that automatic driving systems can stabilize traffic flow by utilizing so-called ITS, where each vehicle can obtain the information of many other vehicles. Our result indicates that there is a limitation in velocity or density of vehicles even in such situation.

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